KSU CET UNIT

FIRST YEAR NOTES

CET UNIT

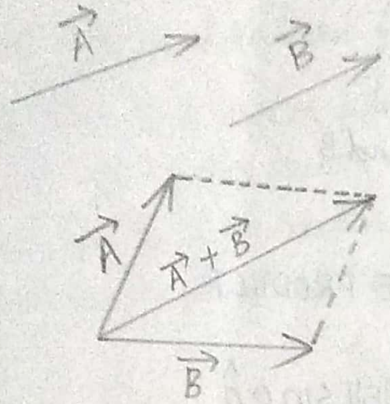
MODULE - I

24/01/19
Friday

VECTOR CALCULUS

Vector - Quantity which has both magnitude and direction
Scalar - Quantity which has only magnitude
Null Vector - Vector with magnitude zero
unit Vector - Vector with magnitude one ($\hat{i}, \hat{j}, \hat{k}$)

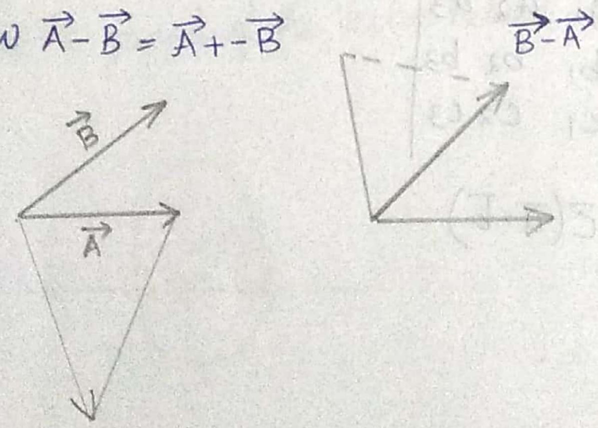
VECTOR ADDITION



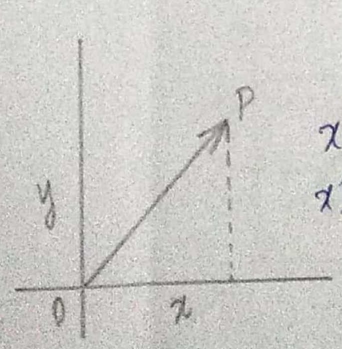
SCALAR MULTIPLICATION OF A VECTOR

A is a vector, c is a scalar $\overrightarrow{CA}$ is in the direction of A with magnitude c times magnitude of A

Draw $\vec{A} - \vec{B} = \vec{A} + -\vec{B}$



POSITION VECTOR OF A POINT



$x\hat{i} + y\hat{j} : (x, y)$

$x\hat{i} + y\hat{j} + z\hat{k} : (x, y, z)$

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\overrightarrow{OP} + \overrightarrow{OQ} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

$c\,\overrightarrow{OP} = cx_1\hat{i} + cy\hat{j} + cz\hat{k}$

DOT PRODUCT OR SCALAR PRODUCT

$\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$A \cdot A = x_1^2 + y_1^2 + z_1^2 = |A|^2$

$A \cdot B = |A||B| \cos\theta$

CROSS PRODUCT

$\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$A \times B = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = |A||B| \sin\theta\, \hat{n}$$

$\hat{n}$ = unit vector ⊥r to A and B

SCALAR TRIPPLE PRODUCT / VECTOR TRIPPLE PRODUCT

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

or $\vec{A} \times \vec{B} = \|A\| \|B\| \sin\theta\, \hat{n}$

where $\hat{n}$ is a unit vector along the direction of ⊥r to A and B

$$[\bar{a}, \bar{b}, \bar{c}] = (\bar{a} \times \bar{b}) \cdot \bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b})$$

Functions :- $|x|$ $x \in (-\infty, \infty)$

$\sin x$

$\cos x$

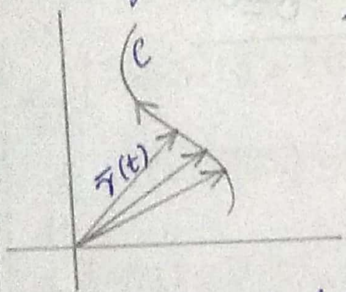
$e^x \longrightarrow$

x^2

VECTOR VALUED FUNCTIONS

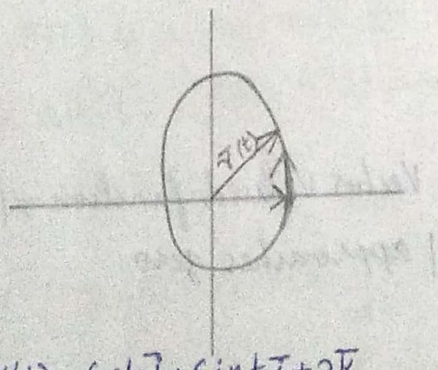
A function which gives a vector is called Vector valued function $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$. Here the components $x(t)$, $y(t)$ and $z(t)$ are real valued function

Let $r(t)$ be a vector whose initial point is at origin
As 't' varies, the vector valued function $r(t)$ will also varies.
Then the tip of the vector $r(t)$ will trace out a curve 'c', It is called graph of the vector valued function $r(t)$ as 't' varies.

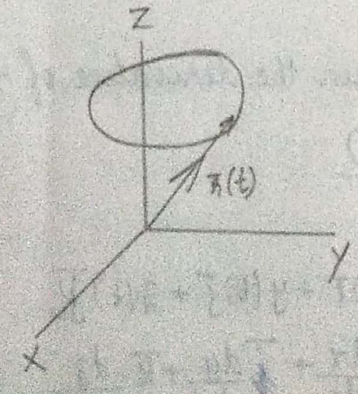


The direction of the graph will be in the increasing direction of 't'.

Draw the graph of the vector valued function $\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j}$

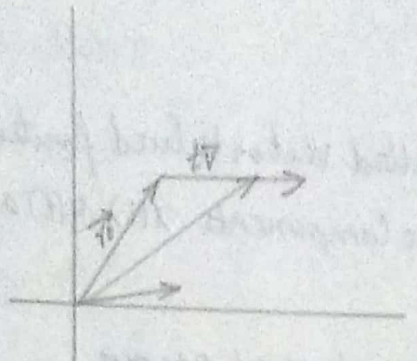


$r(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + 2\bar{k}$

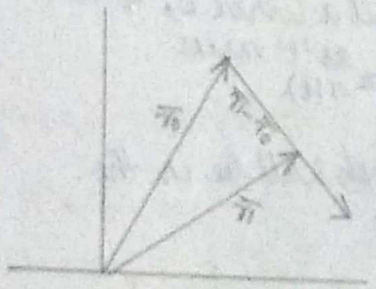


VECTOR FORM OF A LINE SEGMENT

The Vector form of a line segment passing through $\bar{r}_0$ and parallel to $\bar{v}$ is given by $\bar{r}(t) = \bar{r}_0 + t\bar{v}$



Equation of a line segment passing through $\bar{r}_0$ and $\bar{r}_1$



$$\bar{r}(t) = \bar{r}_0 + t(\bar{r}_1 - \bar{r}_0)$$
$$\bar{r}(t) = (1-t)\bar{r}_0 + t\bar{r}_1 \quad , \quad 0 \le t \le 1$$

CALCULUS OF VECTOR VALUED FUNCTION

$y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Limit and Continuity

A vector 'L' is said to be the limit of a vector valued function $r(t)$ at 'a'. If $\|\bar{r}(t) - \bar{L}\|$ is negligible when $|t-a|$ approaches zero

ie, $\bar{L} = \lim_{t \to 0} \bar{r}(t)$

Derivatives

Let $\bar{r}(t)$ be a vector valued function then the derivative of $\bar{r}(t)$ w.r.t 't' is defined as $\bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h}$

$\frac{d\bar{r}(t)}{dt}, \frac{d\bar{r}}{dt}, \bar{r}'(t)$

If $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

$$\frac{d\bar{r}}{dt} = \bar{i}\frac{dx}{dt} + \bar{j}\frac{dy}{dt} + \bar{k}\frac{dz}{dt}$$

$r(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + 2\bar{k}$

$\frac{dr}{dt} = -\sin t\,\bar{i} + \cos t\,\bar{j}$

- $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

$\bar{r}'(t), \frac{d\bar{r}}{dt}$

$\bar{r}'(t) = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$

1. $\frac{d\bar{c}}{dt} = 0$

2. $\frac{d}{dt}a\bar{r}(t) = a\frac{dr}{dt}$

3. $\frac{d}{dt}(\bar{r}(t) + \bar{v}(t)) = \frac{d\bar{r}}{dt} + \frac{d\bar{v}}{dt}$

4. $\frac{d}{dt}(\bar{r}(t) - \bar{v}(t)) = \frac{d\bar{r}}{dt} - \frac{d\bar{v}}{dt}$

5. $\frac{d}{dt}(f(t)\bar{r}(t)) = f(t)\frac{d\bar{r}}{dt} + \frac{df}{dt}\cdot\bar{r}(t)$ $\quad uv = uv' + u'v$

$\frac{u}{v} = \frac{u'v - uv'}{v^2}$

6. $\frac{d}{dt}(\bar{r}(t)\,\bar{v}(t)) = \bar{r}(t)\cdot\frac{d\bar{v}}{dt} + \frac{d\bar{r}}{dt}\cdot\bar{v}(t)$

7. $\frac{d}{dr}(\bar{r}(t) \times \bar{v}(t)) = \bar{r}(t) \times \frac{d\bar{v}}{dt} + \frac{d\bar{r}}{dt} \times \bar{v}(t)$

If $\bar{r}(t)$ is a differentiable vector valued function, $\|\bar{r}(t)\|$ is a constant for all 't'. then $\bar{r}(t)$ and $\bar{r}'(t)$ are orthogonal ($\perp$'r)

ie, $\bar{r}(t)\cdot\bar{r}'(t) = 0$

$\frac{d}{dt}(\bar{r}(t)\cdot\bar{r}(t)) = \bar{r}(t)\cdot\frac{d\bar{r}}{dt} + \frac{d\bar{r}}{dt}\cdot\bar{r}(t)$

$\frac{d}{dt}(\|\bar{r}(t)\|^2) = 2(\bar{r}(t)\cdot\bar{r}'(t))$

$\|\bar{r}(t)\|$ is constant $\Rightarrow \frac{d}{dt}\|\bar{r}(t)\| = 0$

$0 = 2(\bar{r}(t)\cdot\bar{r}'(t))$

ie, $\bar{r}(t)\cdot\bar{r}'(t) = 0$ {orthogonal}

NOTE:-

Let 'P' be a point on the graph of a vector valued function $\bar{r}(t)$.

Let $\bar{r}(t_0)$ be the position vector of 'P'. $\bar{r}'(t)$ exists at $\bar{r}'(t_0) \neq 0$.

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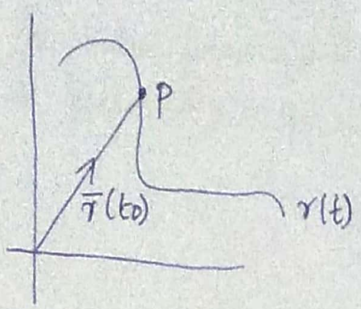
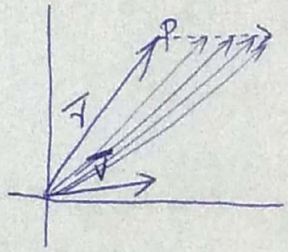
Then we say $\overline{r}'(t_0)$ is the tangent vector to the graph at $\overline{r}(t_0)$



EQUATION OF TANGENT LINE

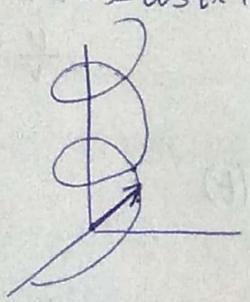


$$\underline{\underline{\overline{r}(t) = \overline{r} + t\overline{v}}}$$

$\boxed{\overline{r}(t) = \overline{r}(t_0) + t\overline{r}'(t_0)}$ $\Rightarrow$ equation of a tangent line

Find the equation of a tangent line to the graph of the vector valued function $\overline{r}(t) = \cos t\,\overline{i} + \sin t\,\overline{j} + t\,\overline{k}$ at $t = \pi$.

$\cos t\,\overline{i} + \sin t\,\overline{j}$
$\Downarrow$
is a circle



Eqn of tangent line $= \overline{r}(t) = \overline{r}(t_0) + t\overline{r}'(t_0)$

$t_0 = \pi$

$\overline{r}(t_0) = \cancel{\cos\pi}\ \overline{r}(\pi)$

$\cancel{\overline{r}'(t_0)} = \cos\pi\,\overline{i} + \sin\pi\,\overline{j} + \pi\,\overline{k}$

$= (-1)\overline{i} + 0\overline{j} + \pi k$

$= \underline{\underline{-\overline{i} + \pi k}}$

$\overline{r}'(t_0) = \overline{r}'(\pi)$

$= -\sin t\,i + \cos t\,j + \overline{k}$

$= 0\overline{i} + (-1)\overline{j} + \overline{k}$

$= \underline{\underline{-\overline{j} + \overline{k}}}$

$\overline{r}(t) = \overline{r}(\pi) + t\overline{r}'(\pi)$

$= (-\overline{i} + \pi k) + t(-j + \overline{k})$

$= -i - t\overline{j} + t(\pi + 1)\overline{k}$

Let the graph of vector valued function $\overline{r}_1(t)$ and $\overline{r}_2(t)$. Intersect at origin. Find the acute angle b/w tangent lines at the point of intersection where $\overline{r}_1(t) = (\tan^{-1} t)i + \sin t\,\overline{j} + t^2 k$

$\overline{r}_2(t) = (t^2 - t)i + (2t - 2)\overline{j} + \ln(t)\,\overline{k}$

$$\bar{a}\cdot\bar{b} = |a||b|\cos\theta$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|a||b|}$$

$$\theta = \cos^{-1}\left(\frac{\bar{a}\cdot\bar{b}}{|a||b|}\right)$$

$$\theta = \cos^{-1}\left(\frac{\bar{r_1}'\cdot\bar{r_2}'}{\|r_1\|\|r_2\|}\right)$$

Tangent Vector to $\bar{r}_1(t)$ at Origin is $\bar{r}_1'(t)$

$$\bar{r}_1'(t) = \frac{1}{1+t^2}\,i + \cos t\, j + 2tk$$

$$\bar{r}_1'(0) = \underline{\underline{\bar{i} + j}}$$

$t^2 i - t i + 2t j - 2j + \ln t\, k$.

$\bar{r}_2'(t) = \cancel{2ti - i + \frac{1}{t}k}$ tangent vector of $r_2(t)$ at origin $= \bar{r}_2'(t)$

$2ti - i + 2j - \frac{1}{t}k$

$= \cancel{2ti - i + \frac{1}{t}k}$ $\quad \bar{r}_2'(t) = (2t-1)\bar{i} + 2\bar{j} + \frac{1}{t}\bar{k}$

$= \cancel{ti + \frac{1}{t}k + 2j}$ $\quad \bar{r}_2'(0) = -\bar{i} + 2\bar{j} -$

VECTOR INTEGRATION

Let $\vec{r}(t)$ be a vector Valued function

$$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

$$\int_a^b \bar{r}(t)\,dt = \int_a^b x(t)\,dt\,\bar{i} + \int_a^b y(t)\,dt\,\bar{j} + \int_a^b z(t)\,dt\,\bar{k}$$

If $R(t) = \bar{r}'(t)$, then $\int R(t)\,dt = \bar{r}(t) + \bar{c}$

PROPERTIES

$$\int_a^b a\,\bar{r}(t)\,dt = a\int_a^b \vec{r}(t)\,dt$$

$$\int_a^b (\bar{r}(t) + \bar{v}(t))\,dt =$$

$$\int_a^b (\bar{r}(t) - \bar{v}(t))\,dt =$$

Evaluate $\int_0^2 (2t\bar{i} + 3t^2 j)\,dt = \int_0^2 2t\bar{i}\,dt + \int_0^2 3t^2\bar{j}\,dt$

$= \cancel{\left(\frac{t^2}{2}\right)_0^2 \bar{i} + t^3}\ \ 2\left(\frac{t^2}{2}\right)_0^2 \bar{i} + 3\left(\frac{t^3}{3}\right)_0^2 \bar{j}$

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$= (t^2)_0^2 \bar{i} + (t^3)_0^2 j$

$= \underline{4\bar{i} + 8\bar{j}}$

Find $\bar{r}(t)$ given that $\bar{r}'(t) = 3\bar{i} + 2t\bar{j}$ and $\bar{r}(1) = 2i + 5j$

$\bar{r}(t) = \int \bar{r}'(t)\,dt$

$\bar{r}'(t) = 3\bar{i} + 2t\bar{j}$

$\therefore \bar{r}(t) = \int (3\bar{i} + 2t\bar{j})\,dt$

$\bar{r}(t) = 3t\bar{i} + t^2\bar{j} + \bar{c}$

Given $r(1) = 2\bar{i} + 5\bar{j}$

$\therefore r(1) = 3(1)\bar{i} + (1^2)\bar{j} + \bar{c}$

$2\bar{i} + 5\bar{j} = 3i + \bar{j} + (c_1\bar{i} + c_2\bar{j})$

$c_1 = -1$

$c_2 = 4$

$\bar{c} = -i + 4\bar{j}$

$\therefore \bar{r}(t) = (3t\bar{i} + t^2\bar{j}) + (-\bar{i} + 4\bar{j})$

$= \underline{(3t-1)\bar{i} + (t^2+4)\bar{j}}$

$2i = 3i + c_1 i, \quad 5j = j + c_2 j$

$2i - 3i = c_1 i \qquad 5j - j = c_2 j$

$\frac{-1 i^2}{i^2} = c_1 \qquad 4j = c_2 j$

$c_1 = \underline{-1} \qquad \frac{c_2 j^2}{j^2} = 4, \; c_2 = \underline{4}$

Find $\bar{r}'(t)$, where $\bar{r}(t) = \tan^{-1} t\,\bar{i} + t\cos t\,\bar{j} - \sqrt{t}\,\bar{k}$

$\bar{r}(t) = \tan^{-1} t\,\bar{i} + t\cos t\,\bar{j} - \sqrt{t}\,k$

$\bar{r}'(t) = \frac{1}{1+t^2}\bar{i} + t(-\sin t)j + \cos t\,j - \frac{1}{2\sqrt{t}}k$

$\bar{r}'(t) = \underline{\frac{1}{1+t^2}\bar{i} - (t\sin t - \cos t)j - \frac{1}{2\sqrt{t}}k}$

Solve the vector initial value problem for $\bar{y}(t)$
Where $\bar{y}'(t) = \cos t\,\bar{i} + \sin t\,\bar{j}$ where $\bar{y}(0) = i - j$

$\bar{y}'(t) = \cos t\,\bar{i} + \sin t\,j$

$\bar{y}(t) = \int \bar{y}'(t)\,dt$

$\bar{y}'(t) = \cos t\,\bar{i} + \sin t\,j$

$\therefore \bar{y}(t) = \int (\cos t\,\bar{i} + \sin t\,\bar{j})\,dt$

$\bar{y}(t) = \sin t\,\bar{i} + \cos t\,j + c$

$\bar{y}(0) = i - j$

$\bar{y}(0) = -\sin 0 i + \cos 0 j + c$

$= 0i + 1j + c$

$i - j = 0i + 1j +$

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$~~y(t) = (\sin t + 1)i - \cos t j~~$

$\bar{y}(t) = \sin t\,\bar{i} - \cos t\,\bar{j} + \bar{c}$

For $\bar{y}(0) = \sin 0\,\bar{i} - \cos 0\,\bar{j} + \bar{c}$

Given $\bar{y}(0) = i - j$

$-j + \vec{c} = \bar{i} - \bar{j}$

$\bar{c} = \bar{i}$

$\bar{y}(t) = (\sin t + 1)\hat{i} - \cos t\,\hat{j}$

Find the Vector equation of a line tangent to the graph of $\bar{r}(t)$ at the point (P_0) on the curve

1. $\bar{r}(t) = (2t-1)i + \sqrt{3t+4}\,j$, $P_0(-1,2)$

Equation of a tangent line $= r_1(t) = r(t_0) + t\bar{r}'(t_0)$

Let $\bar{r}_1(t)$ denote eqn of a tangent line then

$r(t) = (2t-1)\bar{i} + \sqrt{3t+4}\,\bar{j}$

$r(t_0) = (2t-1)\bar{i} + \sqrt{3t_0+4}\,\bar{j}$ $\quad P_0 = (-1, 2)$

$(-i+2j) = (2t_0-1)i + \sqrt{3t_0+4}\,j$

$\left.\begin{array}{l} 2t_0 - 1 = -1 \\ \sqrt{3t_0+4} = 2 \end{array}\right\}$ $t_0 = 0$ $\longrightarrow$ $2t_0 = 0$, $t_0 = 0$ $\quad \sqrt{3t_0+4} = 2$, $3t_0 + 4 = 4$, $3t_0 = 0$, $t_0 = 0$

$r'(t_0) = -i + 2j$

$\bar{r}'(t) = 2\bar{i} + \frac{1}{2\sqrt{3t+4}}3\bar{j} = 2\bar{i} + \frac{3}{2\sqrt{3t+4}}\bar{j}$

$\bar{r}'(t_0) = 2\bar{i} + \frac{3}{3\sqrt{0+4}}\bar{j} = 2i + \frac{3}{4}\bar{j}$ So, $r_1(t) = r(t_0) + t r'(t_0)$

$r_1(t) = -\bar{i} + 2\bar{j} + t\left[2\bar{i} + \frac{3}{4}\bar{j}\right]$

$r_1(t) = (-1+2t)\bar{i} + \left(2 + \frac{3}{4}t\right)\bar{j}$

2. $\bar{r}(t) = t^2\bar{j} - \frac{1}{t+1}\bar{j} + (4-t^2)\bar{k}$, $P_0(4i+j)$

3. $\bar{r}_1(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + t\bar{k}$. $\bar{r}_2(t) = i + t\bar{k}$

S.T. $\frac{d}{dt}(\bar{r}_1(t) \cdot r_2(t)) = \bar{r}_1(t) \cdot \frac{dr_2}{dt} + \frac{dr_1(t)}{dt} \cdot \bar{r}_2(t)$

$\frac{d}{dt}(r_1(t) \times r_2(t)) = \bar{r}_1(t) \times \frac{dr_2}{dt} + \frac{d\bar{r}_1}{dt} \times \bar{r}_2(t)$

2. $\vec{r}(t) = t^2\hat{i} - \frac{1}{t+1}\hat{j} + (4-t^2)\hat{k}$ $\quad P_0 = 4\hat{i}+\hat{j}$

$\bar{r}_1(t) = \bar{r}(t_0) + t\,\bar{r}'(t_0)$ is the eqn of tangent line

$r(t) = t^2\bar{i} - \frac{1}{t+1}\bar{j} + (4-t^2)\bar{k}$

$r(t_0) = t_0^2\bar{i} - \frac{1}{t_0+1}\bar{j} + (4-t_0^2)\bar{k}$

$\Rightarrow 4\bar{i}+j = t_0^2\bar{i} - \frac{1}{t_0+1}\bar{j} + (4-t_0^2)\bar{k}$

$t_0^2 = 4$ or $\frac{-1}{t_0+1} = 1$ or $4-t_0^2 = 0$

$t_0 = \pm 4$ $\quad -1 = t_0+1$ $\quad t_0^2 = 4$

$t_0 = -2$ $\quad t_0 = \pm 2$

$r'(t) = 2t\bar{i} - \left(-\frac{1}{(t+1)^2}\bar{j}\right) + (-2t)\bar{k}$

$r'(t) = 2t\bar{i} + \frac{1}{(t+1)^2}\bar{j} - 2t\bar{k}$

$r'(t_0) = -4\bar{i} + \frac{1}{(-2+1)^2}\bar{j} + 4\bar{k}$

$= -4\bar{i}+\bar{j}+4\bar{k}$

So, $\bar{r}_1(t) = \bar{r}(t_0) + t\,\bar{r}'(t_0)$

$\bar{r}(t) = 4\bar{i}+\bar{j}+t(-4\bar{i}+j+4k)$

$= (4-4t)\bar{i} + (1+t)\bar{j} + 4t\bar{k}$

3. $\bar{r}_1(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + t\bar{k}$

$\bar{r}_2(t) = \bar{i} + t\bar{k}$

$\bar{r}_1(t)\cdot\bar{r}_2(t) = (\cos t\,\bar{i} + \sin t\,\bar{j} + t\bar{k})\cdot(\bar{i}+t\bar{k})$

$\bar{r}_1(t)\cdot\bar{r}_2(t) = \cos t + 0 + t^2 = \cos t + t^2$

$\therefore \frac{d}{dt}(\cos t + t^2) = -\sin t + 2t = LHS$

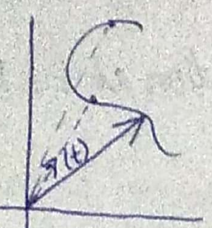
$\bar{r}_1(t)\cdot\frac{d\bar{r}_2(t)}{dt} + \frac{d\bar{r}_1(t)}{dt}\cdot\bar{r}_2(t) = (\cos t\,\bar{i} + \sin t\,\bar{j} + t\bar{k})\cdot(\bar{k}) + (-\sin t\,\bar{i} + \cos t\,\bar{j} + \bar{k})\cdot(\bar{i}+t\bar{k})$

$RHS = (0+0+t) + (-\sin t + 0 + t)$

$RHS = -\sin t + 2t$ $\quad \therefore LHS = RHS$

~~$r_1(t) \times r_2(t)$~~ $r_1(t) \times r_2(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & t \\ 1 & 0 & t \end{vmatrix}$

MOTION ALONG A CURVE



→ Displacement
→ Distance travelled
→ Velocity
→ Speed
→ Acceleration

Displacement $= r(t_2) - r(t_1)$

Velocity $= \frac{d\bar{r}}{dt}$, Speed $= \left\|\frac{d\bar{r}}{dt}\right\|$

Let the motion of a particle is displaced by a smooth vector valued function $r(t)$. $r(t)$ is called the position function of a particle. As the particle moves along a trajectory, its direction of motion and speed varies. then its instanteneous velocity is $\bar{v}(t) = \frac{d\bar{r}}{dt}$

instanteneous acceleration is $\bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

and speed $= \|v\| = \left\|\frac{d\bar{r}}{dt}\right\|$

If the position function $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

then $\bar{v}(t) = \frac{d\bar{r}}{dt}$

$$\bar{v}(t) = \frac{d\left(x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}\right)}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$\bar{a}(t) = \frac{d^2x}{dt^2}\bar{i} + \frac{d^2y}{dt^2}\bar{j} + \frac{d^2z}{dt^2}\bar{k}$$

$$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

DISPLACEMENT AND DISTANCE TRAVELLED

Let $\bar{r}(t)$ be the position function of a moving particle with $\bar{v}(t) = \frac{d\bar{r}}{dt}$

Then its displacement from t_1 to t_2 is given by $\Delta\bar{r} = \int_{t_1}^{t_2} \bar{v}(t)\,dt$

$$\Delta\bar{r} = \left[r(t)\right]_{t_1}^{t_2}$$

$$\Delta\bar{r} = \bar{r}(t_2) - \bar{r}(t_1)$$

distance travelled, $s = \int_{t_1}^{t_2} \|v(t)\|\,dt$

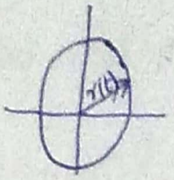
$$s = \int_{t_1}^{t_2} \left\|\frac{dr}{dt}\right\| dt$$

A particle moves along a path with position function $\bar{r}(t) = 2\cos t\,\bar{i} + 2\sin t\,\bar{j}$

(1) find its instantaneous Velocity, Speed and acceleration

(2) find its instantaneous Velocity, speed and acceleration at $t = \pi/4$

$\bar{r}(t) = 2\cos t\,\bar{i} + 2\sin t\,\bar{j}$



$\text{Velocity} = \bar{v}(t) = \dfrac{d\bar{r}}{dt}$

$= \dfrac{d(2\cos t\,\bar{i} + 2\sin t\,\bar{j})}{dt}$

$= -2\sin t\, i + 2\cos t\, j$

$\text{Speed} = \|v(t)\|$

$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$= 2\sqrt{\sin^2 t + \cos^2 t}$

$= 2 \times 1 = 2$

$\text{Acceleration} = -2\cos t\, i - 2\sin t\, j \Rightarrow \dfrac{d\bar{v}}{dt} = \dfrac{d^2\bar{r}}{dt^2}$

(ii) At $t = \frac{\pi}{4}$, $\bar{v}(t) = -2\sin \pi/4\, i + 2\cos \pi/4\, j$

$\bar{v}(t) = -2\frac{1}{\sqrt{2}}\bar{i} + 2\frac{1}{\sqrt{2}}\bar{j}$

$\bar{v}(t) = -\sqrt{2}\,\bar{i} + \sqrt{2}\,\bar{j}$

Speed at $(t = \pi/4) = 2$

$\text{acceleration} = \dfrac{d^2 r}{dt^2}$

$= -2\cos \pi/4\, i - 2\sin \pi/4\, \bar{j}$

$= -2\frac{1}{\sqrt{2}}\bar{i} - 2\frac{1}{\sqrt{2}}\bar{j}$

$= -\sqrt{2}\,\bar{i} - \sqrt{2}\,\bar{j}$

A particle moves with a Velocity $V(t) = i + t\bar{j} + t^2 k$ find the co-ordinates of the particle at $t = 1$ given that the particle was at $(-1, 2, 4)$ initially.

$V(t) = i + t\bar{j} + t^2 k$

$V(t) = \dfrac{d\bar{r}}{dt}$

$\bar{r}(t) = \text{Position function}$

$\bar{r}(t) = \int V(t)\,dt$

$\bar{r}(t) = \int (i + tj + t^2 k)\,dt$

$\bar{r}(t) = \left(t\bar{i} + \frac{t^2}{2} j + \frac{t^3}{3} k\right) + (c_1\bar{i} + c_2\bar{j} + c_3\bar{k})$

$\bar{r}(t) = (t + c_1)\bar{i} + \left(\frac{t^2}{2} + c_2\right)\bar{j} + \left(\frac{t^3}{3} + c_3\right)\bar{k}$

Given when $t = 0$, $\bar{r}(0) = -i + 2j + 4K$

$\bar{r}(0) = (0 + c_1)\bar{i} + \left(\frac{0^2}{2} + c_2\right)\bar{j} + \left(\frac{0^3}{3} + c_3\right)\bar{k}$

$-i + 2j + 4K = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

$c_1 = -1,\ c_2 = 2,\ c_3 = 4$

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$\therefore \bar{r}(t) = (t-1)\bar{i} + \left(\frac{t^2}{2}+2\right)\bar{j} + \left(\frac{t^3}{3}+4\right)k$ at $t=1$, $\bar{r}(t) = 5/2\,\bar{j} + \frac{13}{3}\bar{k}$

$$\bar{r}(t) = 5/2\,\bar{j} + 13/3\,\bar{k}$$

Let a particle moves along a circular helix $\bar{r}(t) = (4\cos\pi t)\bar{i} + (4\sin\pi t)\bar{j} + t\bar{k}$ find distance travelled and displacement. $1 \le t \le 5$

Distance travelled, $S = \int_{t_1}^{t_2} \left\|\frac{dr}{dt}\right\| dt$

Displacement, $\Delta r = \int_{t_1}^{t_2} \frac{dr}{dt}\, dt$

$$\frac{dr}{dt} = -4\pi\sin\pi t\,\bar{i} + 4\pi\cos\pi t\,j + \bar{k}$$

$$\left\|\frac{dr}{dt}\right\| = \sqrt{16\pi^2\sin^2\pi t + 16\pi^2\cos^2\pi t + 1}$$

$$\left\|\frac{dr}{dt}\right\| = \sqrt{16\pi^2+1}$$

$$S = \int_1^5 \sqrt{16\pi^2+1}\, dt$$

$$= \sqrt{16\pi^2+1}\,(t)_1^5$$

$$= 4\sqrt{16\pi^2+1}$$

$$\frac{d\bar{r}}{dt} = (-4\sin\pi t)\pi\bar{i} + (4\cos\pi t)\pi\bar{j} + k$$

$$\frac{d\bar{r}}{dt} = -4\pi\sin\pi t\,\bar{i} + 4\pi\cos\pi t\,\bar{j} + \bar{k}$$

$$\left\|\frac{dr}{dt}\right\| = \sqrt{(-4\pi\sin\pi t)^2 + (4\pi\cos\pi t)^2 + 1^2}$$

$$= \sqrt{16\pi^2[\sin^2\pi t + \cos^2\pi t] + 1}$$

$$= \sqrt{16\pi^2+1}$$

$$\therefore S = \int_1^5 \left(\sqrt{16\pi^2+1}\right) dt$$

$$S = \sqrt{16\pi^2+1}\,(t)_1^5 = 4\sqrt{16\pi^2+1}$$

$$\Delta r = \bar{r}(t_2) - \bar{r}(t_1)$$

$$\Delta r = r(5) - \bar{r}(1)$$

$$\Delta r = 4\cos\pi\, i + 4\sin 5\pi + 5\bar{k} - 4\cos\pi\bar{i} - 4\sin\pi\bar{j} - k$$

$$\Delta r = 4(\cos 5\pi - 4\cos\pi)\bar{i} + (4\sin 5\pi - 4\sin\pi)\bar{j} + 4\bar{k}$$

$$\Delta r = \left[4(-1)^n - 4(-1)^n\right]\bar{i} + (0-0)\bar{j} + 4\bar{k}$$

$$\Delta r = (-4+4)\bar{i} + 0j + 4\bar{k} = \underline{4\bar{k}}$$

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NORMAL AND TANGENTIAL DIFFERENTIATION

Let a particle moves in a plane with a position function $\bar{r}(t) = t^2\bar{i} + \frac{1}{3}t^3\bar{j}$ where t is the time. find the displacement and distance travelled from $t=1$ to $t=3$?

$$\text{displacement} = \cancel{\int_1^3 2t + t^2 = 2} \Big[$$

$$\text{Speed} = \left\| \frac{d\bar{r}}{dt} \right\| =$$

$$\text{Distance travelled} = \int_{t_1}^{t_2} \left\| \frac{dr}{dt} \right\| dt$$

$$= \int_1^3 \sqrt{4t^2 + t^4}\, dt$$

put $u = 4 + t^2$

$du = 2t\,dt$

$t\,dt = \frac{du}{2}$

$$= \int_1^3 t\sqrt{4+t^2}\, dt$$

$$= \cancel{\int_1^3 \sqrt{u}\,\frac{du}{2}} \quad \int t\sqrt{4+t^2}\,dt = \int \sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\,\frac{2}{3}(u)^{3/2}$$

$$= \frac{1}{3}(4+t^2)^{3/2} \qquad = \int_5^{13} \sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\left[\frac{u^{2/3}}{2/3}\right]_5^{13}$$

$$= \frac{1}{2}\left[\frac{(4+13)^{2/3}}{\frac{2}{3}} - \right.$$

$$\text{Velocity} = \frac{d\bar{r}}{dt}$$

$$\text{velocity} = \frac{d\,(t^2\bar{i} + \frac{1}{3}t^3\bar{j})}{dt}$$

$$\text{velocity} = 2t + \frac{1}{3}\times 3t^2$$

$$= 2t + t^2$$

$$\text{displacement} = \int_{t_1}^{t_2} \frac{dr}{dt}\,dt$$

$$= \left[r(t)\right]_{t_1}^{t_2}$$

$$= r(t_2) - r(t_1)$$

$$8\bar{i} + \frac{26}{3}\bar{j}$$

$$\text{Displacement} = \int_1^3 \frac{dr}{dt}\,dt$$

$$\frac{dr}{dt} = \left[2t\bar{i} + \frac{1}{3}\times 3t^2\bar{j}\right]$$

$$\frac{dr}{dt} = 2t\bar{i} + t^2\bar{j}$$

$$\Delta r = \int_1^3 2t\bar{i} + t^2\bar{j}\,dt$$

$$\Delta r = \left[\frac{2t^2}{2}\bar{i} + \frac{t^3}{3}\bar{j}\right]_1^3$$

$$\Delta r = 9i + \frac{27}{3}j - \left(1\bar{i} + \frac{1}{3}\bar{j}\right)$$

$$\Delta r = 8i + \frac{26}{3}\bar{j}$$

Find the position Vector of the particle at time 't' which is moving with an acceleration $\bar{a}(t) = -\cos t\,\bar{i} - \sin t\,\bar{j}$, $\bar{V}(0) = \bar{i}$

$\bar{r}(0) = \bar{j}$

$\bar{a} = \frac{d\bar{v}}{dt}$ $\quad \bar{V} = \int \bar{a}\, dt$

$\bar{V} = \frac{d\bar{r}}{dt}$ $\quad \bar{V} = \int(-\cos t\,\bar{i} - \sin t\,\bar{j})dt$

$\bar{V} = (\sin t\, i + \cos t\, j) + (C_1\bar{i} + C_2\bar{j})$

Given $\bar{V}(0) = i$

$\bar{V}(0) = \sin 0\,\bar{i} - \cos 0\,\bar{j} + C_1\bar{i} + C_2\bar{j}$

$i = (0 + C_1)i + (C_2 + 1)j$

$i = C_1$

$0 = C_2 + 1$

ie, $C_1 = 1$, $C_2 = -1$

$\therefore \bar{V}(t) = (-\sin t + 1)\bar{i} + (\cos t - 1)\bar{j}$

$\therefore \bar{r}(t) = \int \bar{V}(t)dt$

$\bar{r}(t) = \int(-\sin t + 1)\bar{i} + (\cos t - 1)\bar{j}\, dt$

$= \left[(\cos t + t)\bar{i} + (\sin t + t)\bar{j} + C_1\bar{i} + C_2\bar{j}\right]$

$\vec{r}(0) = \bar{j}$

$\therefore 1 + C_1 = 0 \Rightarrow C_1 = -1$; $C_2 = 1$, $\vec{r}(t) = [\cos t + t - 1]\bar{i} + [\sin t - t + 1]\bar{j}$

Find the position vector of the particle at the time 't' which is moving with an acceleration $\bar{a}(t) = \sin t\,\bar{i} + \cos t\,\bar{j} + e^t\,\bar{k}$

$V(0) = \bar{k}$

$r(0) = -\bar{i} + \bar{k}$

TANGENTIAL AND NORMAL COMPONENT OF ACCELERATION

Let $\bar{r}(t)$ be a position vector of moving particle and $\bar{V}(t)$ and $\bar{a}(t)$ be velocity and acceleration. Then $\bar{a}(t)$ can be expressed as $\boxed{\bar{a}(t) = a_T\bar{T} + a_N\bar{N}}$, $\bar{T}$ and $\bar{N}$ are unit vectors along tangent and normal.

$a_T \rightarrow$ tangential scalar component of acceleration

$a_N \rightarrow$ normal scalar component of acceleration

$$\boxed{a_T = \frac{\bar{a}\cdot\bar{v}}{\|V\|}} \qquad \boxed{a_N = \frac{\|\bar{a}\times\bar{v}\|}{\|V\|}}$$

Let a particle moves with a position vector $\bar{r}(t) = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

1. find the Scalar tangential and normal component of acceleration at time, t
2. find the Scalar tangential and normal component of acceleration at time $t=1$
3. find the Vector tangential and normal component of acceleration

$$\bar{a}(t) = \underbrace{a_T\bar{T}}_{V} + a_N\bar{N}$$

normal vector component

$$T = \frac{V}{\|V\|}$$

$$a_N\bar{N} = \bar{a}(t) - a_T\bar{T}$$

$$\bar{r}(t) = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$V(t) = \frac{dr}{dt}$$
$$= i + 2tj + 3t^2k$$

$$a(t) = 2\bar{j} + 6t\bar{k}$$

Scalar tangential component of acceleration, $a_T = \frac{\bar{a}\cdot\bar{v}}{\|V\|}$

$$\|\bar{v}\| = \sqrt{1+4t^2+9t^4}$$

$$\bar{a}\cdot\bar{v} = 4t + 18t^3$$

$$\therefore a_T = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}$$

$$a_N = \frac{\|\bar{a}\times\bar{v}\|}{\|V\|}$$

$$\bar{a}\times\bar{v} = \begin{vmatrix} i & j & k \\ 0 & 2 & 6t \\ 1 & 2t & 3t^2 \end{vmatrix}$$

$$= i\left[(2t\times 3t^2) - (2t\times 6t) - j(-6t) + \hat{k}(-2)\right)$$

$$= i\left[6t^3 - 12t^2\right] + 6t\bar{j} - 2\bar{k}$$

$$= 6t^3\bar{i} - 12t^2 i + 6t\bar{j} - 2\bar{k}$$

$$= i(6t^2 - 12t^2) - j(0-6t) + k(0-2)$$

$$= -6t^2 i + 6tj - 2k$$

$$\|\bar{a}\times\bar{v}\| = \sqrt{36t^4+36t^2-4}$$

$$a_N = \frac{\sqrt{36t^4+36t^2-4}}{\sqrt{1+4t^2+9t^4}}$$

(ii) a_T at $t=1$

$$a_T = \frac{4+18}{\sqrt{1+4+9}} = \frac{22}{\sqrt{14}} = 5.879$$

$$a_N = \frac{\sqrt{36+36+4}}{\sqrt{9+4+1}} = \frac{\sqrt{76}}{\sqrt{14}} = 2.329$$

$$\bar{v}(t) = \int \bar{a}(t)\,dt$$

$$= \int (\sin t\,\bar{i} + \cos t\,\bar{j} + e^t\bar{k})\,dt$$

$$= (-\cos t\,\bar{i} + \sin t\,\bar{j} + e^t\bar{k}) + (c_1 i + c_2 j + c_3 k)$$

$\bar{v}(0) = \bar{k}$

$$\bar{v}(0) = (-\cos 0\,\bar{i} + \sin 0\,\bar{j} + e^0\bar{k}) + (c_1\bar{i} + c_2\bar{j} + c_3\bar{k})$$

$$= (-1\bar{i} + 0\bar{j} + 1k) + (c_1 i + c_2 j + c_3 k)$$

$$\bar{k} = (c_1 - 1)i + (c_2 + 0)j + (c_3 + 1)\bar{k}$$

$c_1 - 1 = 0$, $c_1 = \underline{1}$

$c_2 + 0 = 0$, $c_2 = 0$

$e^t + c_3 = 1$

$e^0 + c_3 = 1$

$c_3 = 1 - 1$

$c_3 = \underline{0}$

$$\bar{v}(t) = (-\cos t + 1)\bar{i} + \sin t\,\bar{j} + e^t\bar{k}$$

$$\bar{r}(t) = \int \bar{v}(t)\,dt = \int ((-\cos t + 1)\bar{i} + \sin t\,\bar{j} + e^t\bar{k})\,dt$$

$$= (-\sin t + t)\bar{i} + (-\cos t)\bar{j} + e^t\bar{k} + c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$$

$$= (-\sin t + t + c_1)\bar{i} + (-\cos t + c_2)\bar{j} + (e^t + c_3)\bar{k}$$

When $\bar{r}(0) = -\bar{i} + \bar{k}$

$\Rightarrow -\sin 0 + 0 + c_1 = -1$

$c_1 = -1$

$\Rightarrow -\cos 0 + c_2 = 0$

$-1 + c_2 = 0$

$c_2 = \underline{1}$

$\Rightarrow e^t + c_3 = 1$

$e^0 + c_3 = 1$

$c_3 = 1 - e^0 = 1 - 1$

$c_3 = \underline{0}$

$$\therefore \bar{r}(t) = (-\sin t + t - 1)\bar{i} + (-\cos t + 1)\bar{j} + (e^t)\bar{k}$$

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DIRECTIONAL DERIVATIVES AND GRADIENTS

Let $f(x,y,z)$ is differentiable at (x_0, y_0, z_0) and $\bar{u} = u_1\bar{i} + u_2\bar{j} + u_3\bar{k}$ be a unit vector
then directional derivative of $f_{(x,y,z)}$ at (x_0, y_0, z_0) in the direction
of u is given by $D_{\bar{u}}f = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$

If $\nabla = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \implies$ gradient of f

$$\boxed{D_{\bar{u}}f = \nabla f \cdot \bar{u}}$$

Find the direction derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of unit vector that makes an angle $\pi/3$ with +ve x-axis

$\bar{u} = u_1\bar{i} + u_2\bar{j}$

$f(x,y) = e^{xy}$

$(x_0, y_0) = (-2, 0)$

Directional derivative, $D_{\bar{u}}f = \nabla f \cdot \bar{u}$

$$D_{\bar{u}}f = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$\frac{\partial f}{\partial x}_{(-2,0)} = y \cdot e^{xy} = \underline{0}$$

$$\frac{\partial f}{\partial y}_{(-2,0)} = e^{xy} \cdot x = -2 \times e^{0} = \underline{\underline{-2}}$$

$\bar{u} = \cos \pi/3\,\bar{i} + \sin \pi/3\,\bar{j}$

$\bar{u} = \frac{1}{2}\bar{i} + \sqrt{3}/2\,\bar{j}$

$D_{\bar{u}}f = 0 \times \frac{1}{2} + (-2)\left(\frac{\sqrt{3}}{2}\right)$

$D_{\bar{u}}f = -\sqrt{3}$

Find the direction derivative of $f(x,y,z) = x^2y - yz^3 + z$ at the point $(1,-2,0)$ in the direction of the Vector $2\bar{i} + \bar{j} - 2\bar{k}$

$D_{\bar{u}}f = f_x U_1 + f_y U_2 + f_z U_3$

$f_x = \frac{\partial f}{\partial x} = \cancel{\quad} \; y \times 2x = 2xy - 0 + 0, \; f_{(1,-2,0)} = 2(1)(-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^2 - z^2 \quad f(1,-2,0) = 1^2 - 0^2 = 1$

$f_z = \frac{\partial f}{\partial z} = -y \times 3z^2 + 1 = -2 \times 3 \times 0^2 + 1 = 1$

$a = 2\bar{i} + \bar{j} - 2\bar{k}$

$\bar{u} = \frac{a}{\|a\|}$

$\|a\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

$= \frac{2}{3}i + 1/3\,\bar{j} - \frac{2}{3}\bar{k}$

$D_{\bar{u}}f = f_x U_1 + f_y U_2 + f_z U_3$

$= -4(2/3) + (1)(1/3) + (1)\left(\frac{-2}{3}\right)$

$= -3$

Find the direction derivative of the function $f(x,y,z) = \frac{z-x}{z+y}$ at point $(1,0,-3)$ in the direction of $\bar{a} = -6\bar{i} + 3\bar{j} - 2\bar{k}$.

$D_{\bar{u}}f = f_x U_1 + f_y U_2 + f_z U_3$

$u = \frac{-6\bar{i} + 3\bar{j} - 2\bar{k}}{\sqrt{36+9+4}}$

$u = -6/7\,i + 3/7\,\bar{j} - 2/7\,\bar{k}$

$-8/63$

$f_x = \frac{\partial f}{\partial x} = \frac{1}{z+y}(0-1)$

$= \frac{-1}{z+y}$

$f_x(1,0,-3) = \frac{-1}{-3+0} = \frac{1}{3}$

$f_y = \frac{\partial f}{\partial y} = z-x \frac{\partial}{\partial y}\left(\frac{1}{z+y}\right) = z-x\left(\frac{-1}{(z+y)^2}\right)(0+1) = -\left(\frac{x-z}{(z+y)^2}\right)$

$f_y(1,0,-3) = -\left(\frac{-4}{9}\right) = \frac{4}{9}$

$$f_z = \frac{\partial f}{\partial z} = \frac{(z+y)(1)-(z-x)(1)}{(z+y)^2}$$

$$= \frac{z+y-z+x}{(z+y)^2} = \frac{1}{9}$$

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{-6\bar{i}+3\bar{j}-2\bar{k}}{\sqrt{36+9+4}} = \frac{-6}{7}\bar{i} + \frac{3}{7}\bar{j} - \frac{2}{7}\bar{k}$$

$$D_{\bar{u}}f = \frac{1}{3}\times -\frac{6}{7} + \frac{4}{9}\times\frac{3}{7} + \frac{1}{9}\times -\frac{2}{7} = \frac{-2}{7} + \frac{4}{21} - \frac{2}{63} = -8/63$$

NOTE

$D_{\bar{u}}f = \nabla f \cdot \bar{u}$

$D_{\bar{u}}f = \|\nabla f\| . \|\bar{u}\| \cos\theta$

$D_{\bar{u}}f = \|\nabla f\| \cos\theta$

Where θ is the angle b/w ∇f and $\bar{u}$
hence the direction derivative, $D_{\bar{u}}f$ will be maximum, if ∇f is in the same direction of $\bar{u}$. And the maximum value is $\|\nabla f\|$.

$D_{\bar{u}}f$ will be minimum, if ∇f and $\bar{u}$ are in the opposite direction and the minimum value is $-\|\nabla f\|$

? Find the maximum value of direction derivative at $(-2,0)$ $f(x,y) = x^2 e^y$. Also find a unit vector in that direction

Max $D_{\bar{u}}f$ occurs with $\bar{u}$ at ∇f are into same direction

$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) f$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}$

$\nabla f = \bar{i}\, 2x e^y + \bar{j}\, x^2 e^y$

$\nabla f(2,0) = \bar{i}\left(2(-2)e^0\right) + j(-2)^2 e^0$

$= -4\bar{i} + 4\bar{j}$

$u = \frac{\nabla f}{\|\nabla f\|}$

$\|\nabla f\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

$\therefore \bar{u} = \frac{-4}{4\sqrt{2}}\bar{i} + \frac{4}{4\sqrt{2}}\bar{j}$

$\bar{u} = \frac{-i}{\sqrt{2}} + \frac{\bar{j}}{\sqrt{2}}$

$D_{\bar{u}}f(2,0) = \|\nabla f\| = 4\sqrt{2}$

VECTOR FIELDS

A Vector field is a function, F that associates with each point, $r = (x, y, z)$ a vector $F(r)$

NOTE :- If $\vec{r}$ is a position vector of a point (x, y, z) on a vector field. Then the vector field is denoted by $\vec{F}(r)$ or $\vec{F}$

$$\vec{F} = x^2\bar{i} + xy\bar{j} + x\bar{k}$$

then the operator, $\nabla = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$

If $\vec{F}(x, y, z)$ is a vector field defined by $f_{(x,y,z)}\bar{i} + g(x, y, z)\bar{j} + h(x, y, z)\vec{k}$

Then the divergence of F is defined as $\nabla \cdot \vec{F}$

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}}$$

curl F is defined as $\Rightarrow$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) - j\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) + k\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

? Find the divergence and curl of $F(x, y, z) = x^2y\bar{i} + 2y^3z\bar{j} + 3z\bar{k}$

divergence, $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$$\nabla \cdot \vec{F} = \frac{\partial (x^2y)}{\partial x} + \frac{\partial (2y^3z)}{\partial y} + \frac{\partial (3z)}{\partial z}$$

$$\nabla \cdot \vec{F} = 2xy + 6y^2z + 3$$

$f = x^2y\bar{i}$

$g = 2y^3z$

$h = 3z$

$\frac{\partial f}{\partial x} = x^2y = 2xy$

$\frac{\partial g}{\partial y} = \frac{\partial (2y^3z)}{\partial y} = 6y^2z$

$\frac{\partial h}{\partial z} = \frac{\partial (3z)}{\partial z} = 3$

curl F, $\nabla \times \vec{F} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & g & h \end{vmatrix}$

$\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$, show that

- $\text{Div}\,\bar{r} = 3$
- $\text{Curl}\,\bar{r} = 0$
- $\nabla\|\bar{r}\| = \dfrac{\bar{r}}{\|\bar{r}\|}$
- $\boxed{\nabla \dfrac{1}{\|\bar{r}\|} = \dfrac{-\bar{r}}{\|\bar{r}\|^3}}$

$$\text{Div}\,\bar{r} = \nabla\cdot\bar{r}$$

$$\text{Div}\,\bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xi + yj + zk)$$

$$\text{Div}\,\bar{r} = \left(\frac{\partial i}{\partial x} + \frac{\partial j}{\partial y} + \frac{\partial k}{\partial z}\right)$$

$$\text{Div}\,\bar{r} = 1+1+1 = \underline{3}$$

$$\text{Curl}\,\bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\text{Curl}\,\bar{r} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - j\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$\text{Curl}\,\bar{r} = \underline{\underline{0}}$$

$$\|\bar{r}\| = \sqrt{x^2+y^2+z^2}$$

$$\nabla\|\bar{r}\| = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= i\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + j\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + k\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}$$

$$= i\frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x + j\frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2y + \bar{k}\frac{1}{2\sqrt{x^2+y^2+z^2}} 2z$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}(\bar{i}x + \bar{j}y + \bar{k}z) = \frac{1}{\|\bar{r}\|}\cdot\bar{r} = \underline{\underline{\frac{\bar{r}}{\|\bar{r}\|}}}$$

GRADIENT FIELDS

Let ϕ be a function on three variables then the gradient field of ϕ is a vector field given by $\nabla\phi$.

Eg:- The gradient field of $\phi(x,y) = x+y$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)(x+y)$$

$$\underline{\underline{\nabla\phi = \bar{i} + \bar{j}}}$$

NOTE:- A vector field $\vec{F}$ is said to be conservative, if it is gradient field of some other function. In other words, there is exist a function, $\vec{F} = \nabla\phi$. And ϕ is called potential function of $\vec{F}$.

LINE INTEGRALS

Let 'C' is a Smooth Curve parametrized by $x(t)\bar{i}+y(t)\bar{j}=\bar{r}(t)$, $a\le t\le b$ and $f(x,y)$ be a function on x and y. Then S be the arc length

$$\int_C f(x,y)\,ds=\int_a^b f\big(x(t)\,y(t)\big)\,\|\bar{r}'(t)\|\,dt$$ and similarly

$x\bar{i}+y\bar{j}+z\bar{k}=\bar{r}(t)$ then $\int_C f(xyz)\,ds=\int_a^b f\big(x(t)\,y(t)\,z(t)\big)\,\|\bar{r}'(t)\|\,dt$

1. Evaluate $\int_C (1+xy^2)\,ds$, where C is the Curve, $C: \bar{r}(t)=t\bar{i}+2t\bar{j}, 0\le t\le 1$

$$\int_C f(xy)\,ds=\int_a^b f\big(x(t),y(t)\big)\,\|\bar{r}'(t)\|\,dt$$

$\bar{r}(t)=t\bar{i}+2t\bar{j}$

$\bar{r}'(t)=\bar{i}+2\bar{j}$

$\|\bar{r}'(t)\|=\sqrt{1+4}=\underline{\underline{\sqrt{5}}}$

$x(t)=t,\ y(t)=2t$

$$\therefore \int_C (1+xy^2)\,ds=\int_0^1 1+t(2t)^2\sqrt{5}\,dt$$

$$=\sqrt{5}\int_0^1 (1+4t^3)\,dt$$

$$=\sqrt{5}\,[t]_0^1+4\sqrt{5}\cdot\left[\frac{t^4}{4}\right]_0^1$$

$$=\sqrt{5}(1-0)+\sqrt{5}(1-0)$$

$$=\underline{\underline{2\sqrt{5}}}$$

2. Evaluate $C: \bar{r}(t)=(1-t)\bar{i}+(2-2t)\bar{j}$; $0\le t\le 1$

$\bar{r}(t)=(1-t)\bar{i}+(2-2t)\bar{j}$

$\bar{r}'(t)=(-1)\bar{i}+(-2)\bar{j}$

$\|\bar{r}'(t)\|=\sqrt{(-1)^2+(-2)^2}=\underline{\sqrt{5}}$

$$x(t)=(1-t)=\int_0^1 \big(1+(1+t)(2-2t^2)\big)\sqrt{5}\,dt$$

$$=\int_0^1 \big(1+(1-t)(4-4t+4t^2)\big)\sqrt{5}\,dt$$

$$=\int_0^1 (1+4-4t+4t^2+4t$$

$\bar{r}(t) = (1-t)r =$
$= (1-t)(i$

? Evaluate $\int_c xyz^2 ds$, where c is a curve represented by parametric eqn $x=t, y=3t^2, z=6t^3, 0 \le t \le 1$

$$\int_C f(xyz)\, ds = \int_{t_1}^{t_2} f(x,y,z)\, \|\vec{r}'\|\, dt$$

$$\bar{r}(t) = t\bar{i} + 3t^2\bar{j} + 6t^3\bar{k}$$

$$\bar{r}' = \frac{d\vec{r}}{dr} = \frac{d(t\bar{i} + 3t^2\bar{j} + 6t^3\bar{k})}{dr} = \bar{i} + 6t\bar{j} + 18t^2 K$$

$$\|\vec{r}'\| = \sqrt{1^2 + 6^2 + 18^2} = \underline{19}$$

$$\int xyz^2 ds = \int_0^1 (t)(3t^2)(6t^3)\, 19\, dt$$

$$= \left[\frac{t^2}{2} \times 3 \times 2t \times 18t^2\right.$$

If $f(xyz)$ is a density function of a wire, 'c' then its mass is given by, mass $= \int_c f(x,y,z)\, ds$

Line integral wrt to x, y and z

Let C be a curve parametrized by $x(t)$, $y(t)$ and $z(t)$ and $f(xyz)$ be a function then ~~integral~~ $\int_c f(xyz)\, dx = \int_a^b f(x(t), y(t), z(t)) \frac{dx}{dt} \cdot dt$

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$$\int_C f(xyz)\,dy = \int_a^b f(x(t), y(t), z(t))\, y'\,dt$$

$$\int_C f(xyz)\,dz = \int_a^b f(x(t), y(t), z(t))\, z'\,dt$$

Evaluate $\int xyz^2\,dx$, $\int xyz^2\,dy$ and $\int_C xyz^2\,dz$ where C is a curve with $x=t,\ y=3t^2,\ z=6t^3,\ 0\le t\le 1$

$f(x,y,z) = xyz^2$

$x=t,\ y=3t^2,\ z=6t^3,\ 0\le t\le 1$

$$\int_C f(x,y,z)\,dx = \int_0^1 f(x(t), y(t), z(t)\ x'(t)\,dt$$

$$x'(t) = \frac{dt}{dt} = \underline{\underline{1}}$$

$$= \int_0^1 (t)(3t^2)(6t^3)^2\,(1)\,dt$$

$=$

$$\int_C f(xyz)\,dy = \int_0^1 x(t), y(t), z(t)\, y'\,dt$$

$$y' = \frac{dy}{dt} = \frac{d(3t^2)}{dt} = \underline{\underline{6t}}$$

$$\int_C xyz^2\,dy = \int_0^1$$

Evaluate $\int_C 3xy\,dy$ over C, where C is a line segment oreinted from $a(0,0)$ $(1,1)$ to $b(1,2)(0,0)$

The parametric eqn for ~~yo~~ line segment joining from r_0 and r_1

$r(t) = (1-t)r_0 + t r_1 \Rightarrow$ (vector eqn of a line from r_0 to r_1)

$a \Rightarrow r_0 = 0i + 0j$

$r_1 = i + 2j$

$\therefore r(t) = (1-t)(0+0) + t(i+2j)$

$= \underline{\underline{ti + 2tj}}$

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Parametric eqn of the line segment from $(0,0)$ to $(1,2)$

$x(t)=t,\ y(t)=2t,\ 0\le t\le 1$

$$\int_C 3xy\,dy = \int_0^1 3(t)(2t)\,y'dt$$

$$= \int_0^1 6t^2\,2\,dt = \int_0^1 12t^2\,dt$$

$$= \frac{12}{3}\left(t^3\right]_0^1 = \underline{4}$$

$$\boxed{\underline{\text{NOTE}}:- \int_C f(x,y,z)\,dx = \int_{-C} f(x,y,z)\,dx}$$

Evaluate $\int_C (3x^2+y^2)\,dx + 2xy\,dy$ where C is a curve, $x=\cos t,\ y=\sin t$, $0\le t\le \pi/2$ in the counterclockwise direction

$$\int_C (3x^2+y^2)\,dx + 2xy\,dy = \int_C (3x^2+y^2)\,dx + \int_C 2xy\,dy$$

$$\int_C (3x^2+y^2)\,dx = \int_0^{\pi/2} f(x,y)\,x'(t)\,dt$$

$x(t)=\cos t,\ y=\sin t,\ 0\le t\le \pi/2$

$f(x,y)=3x^2+y^2$

$$\int_C (3x^2+y^2) = \int_0^{\pi/2} (3\cos^2 t+\sin^2 t)\sin t\,dt$$

$$= \int_0^{\pi/2} (3\cos^2 t+(1-\cos^2 t))\sin t\,dt$$

$$= \int_0^{\pi/2} (2\cos^2 t+1)-\sin t\,dt$$

put $\cos t = u$

$-\sin t\,dt = du$

$t=0,\to u=1$

$t=\pi/2, u=0$

$$\int_1^0 (2u^2+1)\,du = \left[2\frac{u^3}{3}+u\right]_1^0 = 0-\left[\frac{2}{3}+1\right] = \underline{\underline{-5/3}}$$

$$\int_C 2xy\,dy = \int_0^{\pi/2} 2\cos t\sin t\cos t\,dt = \int_0^{\pi/2} 2\cos^2 t\sin t\,dt$$

Put $\cos t = u$ $\quad t=0\to u=1$

$-\sin t\,dt = du$ $\quad t=\pi/2\to u=0$

$$= -\int_1^0 2t^2 du = \left[\frac{2t^3}{3}\right]_0^1 = -\left[0 - 2\times\frac{1^3}{3}\right] = 2/3$$

Evaluate $\int_C x^2 dy - yz\,dz$, where c is a line segment from $(4,-1,2)$ to $(1,7,-1)$?

$\bar{r}(t) = (1-t)\bar{r}_0 + t\bar{r}_1$

$\bar{r}_0 = 4\bar{i} - \bar{j} + 2\bar{k}$

$\bar{r}_1 = i + 7j - k$

$r(t) = (1-t)(4\bar{i} - \bar{j} + 2\bar{k}) + t(i + 7j - k)$

$r(t) = (4 - 4t + t)\bar{i} + (t - 1 + 7t)\bar{j} + (2 - 2t - t)\bar{k}$

$r(t) = (4 - 3t)\bar{i} + (-1 + 8t)\bar{j} + (2 - 3t)\bar{k}$

$x(t) = 4 - 3t$; $y(t) = 8t - 1$, $z(t) = 2 - 3t$

$0 \le t \le 1$

$y'(t) = 8$, $z'(t) = -3$

$$\int_C x^2 dy - yz\,dz = \int_0^1 (4-3t)^2 \cdot 8\,dt - (8t-1)(2-3t)(-3)\,dt$$

$$= \int_0^1 \left[8(4-3t)^2 - 3(1-8t)(2-3t)\right] dt$$

$$= \int_0^1 \left[8(4^2 - 24t + 9t^2) - 3(1-8t)(2-3t)\right] dt$$

$$= \int_0^1 \left[8[16 - 24t + 9t^2) - 3(1-8t)(2-3t)\right] dt$$

$$= \int_0^1 8(16 + 9t^2 - 24t) - 3(2 - 3t - 16t + 24t^2) \quad -19t$$

$$= \int_0^1 128 + 72t^2 - 192t - 6 + 57t - 72t^2$$

$$= \int_0^1 128 - 192t - 6 + 57t$$

$$= \left[122 - 135t\right]_0^1 = \left[122t - \frac{135t^2}{2}\right]_0^1$$

$$= 122 - \frac{135}{2} = \frac{109}{2} = 54.5$$

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$\int_C -y\,dx + x\,dy$, where $C: y^2 - 3x$ from $(3,3)$ to $(0,0)$

Let $y = t$

$x = \frac{t^2}{3}$

$t = 3$ to 0

$\frac{dx}{dt} = \frac{2t}{3}, \frac{dy}{dt} = 1$

$$\int_C -y\,dx + x\,dy = \int_3^0 -t \cdot \frac{2t}{3}\,dt + \frac{t^2}{3} \cdot 1\,dt$$

$$= \int_3^0 \left[\frac{-2t^2}{3} + \frac{t^2}{3}\right] dt$$

$$= \int_3^0 \frac{-1}{3} t^2\,dt = -\frac{1}{3}\left[\frac{t^3}{3}\right]_3^0$$

$$= \frac{-1}{9}\left[0 - 27\right] = \underline{\underline{3}}$$

~~LINE~~ INTEGRATING A VECTOR FIELD ALONG A curve

- $\vec{F}(r) = f(x,y)\,\bar{i} + g(x,y)$

Let F be a continuous vector field and C be a smooth curve then the line integral of F along C is $\int_C \vec{F} \cdot d\vec{r}$, where $d\vec{r} = x\bar{i} + y\bar{j}$

$\vec{r} = xi + yj$

$d\vec{r} = dx\bar{i} + dy\bar{j}$

~~$\int \vec{F} \cdot dr = \int ...$~~

$\rightarrow$ If the vector field $\vec{F}(x,y) = f(x,y)\bar{i} + g(x,y)\bar{j}$

$$\int_C F(x,y) \cdot d\bar{r} = \int_C f(x,y)i + g(x,y)j = (i\,dx + j\,dy)$$

$$= \int_C f(x,y)\,dx + g(x,y)\,dy$$

NOTE :- If the curve, C is parametrized by $\bar{r}(t)$, $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j}$

$a \leq t \leq b$. $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot (\bar{r}'(t))\,dt$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $f(x,y) = \cos x\,\bar{i} + \sin x\,\bar{j}$, where C is the curve $r(t) = \frac{-\pi}{2} i + t\bar{j}$, $1 \leq t \leq 2$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \cancel{F(\bar{r}(t)) \cdot \bar{r}'(t)\,dt} \int F(x,y)\,dx + g(x,y)\,dy$$

$F(x,y) = \cos\bar{i} + \sin x\,\bar{j}$

$f(x,y) = \cos x$

$g(x,y) = \sin x$

$\int_C F \cdot dr = \int_C \cos x\,dx + \sin x\,dy$

$C: r(t) = -\frac{\pi}{2} i + tj$

$x(t) = -\pi/2$; $x'(t) = 0$

$y(t) = t$, $y'(t) = 1$

$$\int_C F \cdot dr = \int_1^2 \cos x \cdot 0 \cdot dt + \int_1^2 \sin x \cdot 1\,dt$$

$$= \int_1^2 \sin x\,dt$$

$$= \sin x\left[t\right]_1^2 = \sin x(2-1)$$

$$= \underline{\underline{\sin x}}$$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is a curve $\vec{F}(x,y) = y^2\vec{i} + (3x-6y)\vec{j}$, C: (3,7) to (0,12)

$\vec{F}(x,y) = y^2\vec{i} + (3x-6y)\vec{j}$

$\int_C \vec{F}\cdot d\vec{r} = \int F(x,y)\,dx + g(x,y)\,dy$

$F(x,y) = y^2$

$g(x,y) = 3x-6y$

$C: \bar{r}(t) = (1-t)\bar{r}_0 + t\bar{r}_1$

$\left.\begin{array}{l} r_0 = 3i+7j \\ r_1 = 12j \end{array}\right\}$ $C: \bar{r}(t) = (1-t)(3i+7j) + (t\,12j)$

$\bar{r}(t) = 3(1-t)i + (7(1-t)+12t)j$

$\bar{r}(t) = (3-3t)\bar{i} + (7+5t)\bar{j}$, $0 \le t \le 1$

$x(t) = 3-3t$

$y(t) = 7+5t$ $0 \le t \le 1$

$\int_C \vec{F}\cdot d\vec{r} = \int_{t_0}^{t_1} F(\bar{r}(t))\cdot \bar{r}'(t)\,dt$

$F(r(t)) = (7+5t)^2 i + (3(3-3t) - 6(7+5t))\bar{j}$

$= (49+70t+25t^2)i + (9-9t-42-30t)\bar{j}$

$= (49+70t+25t^2)i + (-33-39t)\bar{j}$

$\bar{r}(t) = (3-3t)i + (7+5t)\bar{j}$

$\bar{r}'(t) = -3i+5j$

$\int_C \vec{F}\cdot d\bar{r} = \int_0^1 \left[(49+70t+25t^2)i + (-33-39t)j\right]\left[-3i+5j\right]dt$

$= \int_0^1 \left(-3(49+70t+25t^2) + 5(-33-39t)\right)dt$

$= \left[-3\left[49t + \frac{70t^2}{2} + \frac{25t^3}{3}\right] - 5\left[33t + \frac{39t^2}{2}\right]\right]_0^1$

$= -3\left(49+35+\frac{25}{3}\right) - 5\left(33+\frac{39}{2}\right) = -\frac{1079}{2}$

NOTE:- Suppose a particle moves along a smooth curve 'C' in the influence of a continuous force field $\mathbf{F}$ then the work done by the force on the particle will $\int_C \vec{F}\cdot d\vec{r}$

? Find the work done by the force field 'F' on a particle that moves along the curve 'C'. $F(x,y) = xy\bar{i} + x^2\bar{j}$, $C: x = y^2 (0,0) \to (1,1)$

Work done $= \int_C \vec{F}\cdot d\vec{r}$

$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j}$

$\bar{r}(t) = t^2 i + tj$

To parametrize, put $y = t$

$x = t^2, 0 \le t \le 1$

$\bar{r}'(t) = 2ti + 1j$

$\cancel{\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j}}$

$\cancel{\bar{r}(t) = t^2\bar{i} + t\bar{j}}$

$\cancel{\bar{r}'(t) = (1-t)(0\vec{i} + 0\vec{j}) + t(i+j)}$

$\cancel{\bar{r}(t) = (1-t)\vec{r_0} + t(\vec{r_1})}$

$= \cancel{(1-t)\times(0\bar{i} + 0\bar{j}) + t(\bar{i}+\bar{j})}$

$= \cancel{(t\bar{i} + t\bar{j})}$

$\cancel{x(t) = t}$

$\cancel{y(t) = t}$

$\cancel{\bar{r}'(t) = i + j}$

$\cancel{F(r)}$

$\cancel{x(t) = t}$

$\cancel{y(t) = t}$

$\cancel{\int_C \vec{F}\cdot d\vec{r} =}$

Find the work done by the force field F, $F(x,y,z) = xy\bar{i} + yz\bar{j} + xz\bar{k}$ where C is a curve, $C: r(t) = t\bar{i} + t^2 j + t^3 k, 0 \le t \le 1$.

Work done $= \int_C \vec{F}\cdot d\vec{r}$

H:W

1. Find the work done by the force field $F(x,y,z) = (x+y)i + xyj - z^2k$ to move along a curve; C is a line segment from $(0,0,0)$ to $(1,3,1)$ and then to $(2,-1,4)$?

2. Evaluate $\int_C F\cdot dr$, $F = yi + xj$ where C is a curve

a. a line segment from $(0,0)$ to $(1,1)$

b. a Parabola $y = x^2$ from $(0,0)$ to $(1,1)$

c. $y = x^3$ from $(0,0)$ to $(1,1)$.

(a) $\bar{r}(t) = (1-t)\bar{r}_0 + t\bar{r}_1$, $0 \le t \le 1$

$\bar{r}(t) = \cancel{(1-t)(0i+0j+0k) + t(i+3j+k)}\ (1-t)(0i+0j) + t(i+j)$

$\bar{r}(t) = \cancel{t\bar{i} + 3t\bar{j} + t\bar{k}}\ t\bar{i} + t\bar{j}$

$\bar{r}'(t) = \bar{i} + \bar{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t_0}^{t_1} F(\bar{r}(t))\cdot \bar{r}'(t)\,dt$$

$F((\bar{r}'(t))) =$

(c) $y = x^3$

$x = t;\ y = t^3,\ 0 \le t \le 1$

$F(x, y) = (y\mathbf{i} + x\mathbf{j})$

$F \cdot dr = (y\mathbf{i} + x\mathbf{j}) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy)$

$F \cdot dr = y\,dx + x\,dy$

$$\int_C F \cdot dr = \int_C y\,dx + x\,dy$$

$$= \int_0^1 t^3\,dt + t \cdot 3t^2\,dt$$

$$= \int_0^1 4t^3\,dt = \left(\frac{4t^4}{4}\right)_0^1 = 1$$

$dx = \frac{dx}{dt} \cdot dt$

$dx = 1\,dt$

$dy = \frac{dy}{dt}\,dt$

$dy = 3t^2\,dt$